

Ballistic-annihilation kinetics for a multiveLOCITY one-dimensional ideal gas

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(Received 16 December 1994)

Ballistic-annihilation kinetics for a multiveLOCITY one-dimensional ideal gas is studied in the framework of an exact analytic approach. For an initial symmetric three-velocity distribution, the problem can be solved exactly and it is shown that different regimes exist, depending on the initial fraction of particles at rest. Extension to the case of an n -velocity distribution is discussed.

PACS number(s): 05.40.+j, 05.20.Dd, 82.20.-w

I. INTRODUCTION

Ballistically controlled reactions provide simple examples of nonequilibrium systems with complex kinetics. In the one-dimensional case, one considers point particles which move freely, with a given velocity. When two particles collide they instantaneously annihilate each other and disappear from the system. The system with only two possible velocities $+c$ or $-c$ has been studied in a pioneering work by Elskens and Frisch [1]. Using combinatorial analysis, they showed that the density of particles was decreasing according to a power law ($t^{-1/2}$) in the case of a symmetric initial velocity distribution. Later, Krug and Spohn [2] obtained independently similar results. Recently, Redner and co-workers [3,4] have studied (essentially numerically) the problem with more general velocity distributions. However, to our knowledge, no rigorous analytical approach has been elaborated until one of us (J.P.) developed a formalism in which the annihilation dynamics reduced exactly to a single closed equation for the two-particle conditional probability [5]. The purpose of this work is to present a method which permits one to solve this evolution equation for discrete velocity distributions. More precisely, we shall consider the three-velocity case where the initial velocity distribution $\phi(v; t = 0)$ is given by

$$\phi(v; 0) = p_+ \delta(v - c) + p_0 \delta(v) + p_- \delta(v + c), \quad (1)$$

with $p_+ = p_-$ (symmetric case), and $p_+ + p_0 + p_- = 1$.

For the ballistic motion, the collision frequency between two particles is proportional to their relative velocity. Then, collisions involving particles at rest occur less often than the ones involving moving particles. Indeed, the collisions between one particle of velocity $+c$ and one of velocity $-c$ (relative velocity $2c$) occur twice as often as those between a particle at rest and a moving one (relative velocity c). As a consequence, one can expect that in one half of the total number of collisions two

moving particles participate, and the other half involves one particle at rest. Thus, in mean, only one quarter of the particles annihilated in collisions are at rest and the remaining three quarters are formed by moving particles. This simple heuristic argument suggests that the values $p_0^* = \frac{1}{4}$, $p_+^* = p_-^* = \frac{3}{8}$ must play a special role: if p_0 is less than p_0^* , the system will asymptotically behave as in the two-velocity case (the stationary particles will disappear before the annihilation of the moving ones). On the contrary, if p_0 is greater than p_0^* , the moving particles will disappear first, and asymptotically only particles at rest will be left. In the limiting case $p_0 = p_0^*$ the moving particles and the ones at rest disappear at the same rate. It will turn out that this intuitive argument is exact. We shall show it using the approach presented in [5], which permits us to solve this three-velocity model exactly.

This paper is organized as follows. In Sec. II, we introduce the model and recall the main steps of the rigorous method [5]. An important quantity is $S(v; t)$, the survival probability until the time t of a particle moving with velocity v . In Sec. III, we study the behavior of $S(v; t)$ and compute the density and the time dependent velocity distribution in the asymptotic regime $t \rightarrow \infty$. The value $p_0 = \frac{1}{4}$ appears to be a critical point, separating different kinetic behaviors. In addition, it is shown that the dynamics is incompatible with a Boltzmann-like approximation due to the appearance of strong velocity correlations between particles which are nearest neighbors. The case of a general discrete multiveLOCITY distribution is examined in Sec. IV. Finally, concluding remarks are given in Sec. V.

II. THE THREE-VELOCITY MODEL

We assume that initially the particles are uniformly distributed in space, according to the Poisson law, without any correlations between their velocities. Note that other distributions than Poisson could be considered as

long as one is dealing with a renewal process. The process is thus translationally invariant and each particle has initially the same probability density $\phi(v; 0)$ [given by Eq. (1)] to move with velocity v .

The important characteristic of the annihilation dynamics is that only those particles which suffered no collisions are present in the system. They are thus found on their free trajectories. Accordingly, the key quantity of the theory is the survival probability $S(v; t)$ which has the product structure

$$S(v; t) = S^R(v; t) S^R(-v; t), \quad (2)$$

where $S^R(v; t)$ is the probability for the absence of collision with the right neighbor. As $S^R(-c; t) = 1$, for $t \geq 0$, we find

$$S(+c; t) = S(-c; t) = S^R(+c; t), \quad (3)$$

$$S(0; t) = [S^R(0; t)]^2. \quad (4)$$

Initially, $S^R(+c; 0) = S^R(0; 0) = 1$.

The density of particles with velocity v at time t is given by

$$\sigma(v; t) = \sigma S(v; t) \phi(v; 0), \quad (5)$$

where σ is the initial density.

Another important quantity in the approach [5] is the distribution of nearest neighbors. Suppose that at time t there is a particle moving with velocity v . We denote by $\mu(x, u|0, v; t)$ the conditional probability density for finding its nearest neighbor to the right at a distance $x > 0$, with velocity u (for nonhomogeneous systems the probability may depend upon the positions of both particles,

rather than on their relative distance only). As we are dealing here with symmetric initial velocity distribution, the following relation holds for μ :

$$\begin{aligned} \phi(v; 0) S(v; t) \mu(x, u|0, v; t) \\ = \phi(-u; 0) S(-u; t) \mu(x, -v|0, -u; t). \end{aligned} \quad (6)$$

The initial condition is given by

$$\mu(x, u|0, v; t = 0) = \theta(x) \sigma e^{-x\sigma} \phi(u; 0), \quad (7)$$

where $\theta(x)$ is the usual Heaviside unit step function.

We also define the density μ at contact by

$$\mu(0+, u|0, v; t) = \lim_{\substack{x \rightarrow 0 \\ x > 0}} \mu(x, u|0, v; t). \quad (8)$$

This quantity plays a particular role, because it determines the density of precollisional configurations.

The time evolution of $S^R(v; t)$ is given by

$$\begin{aligned} S^R(+c; t) = \exp \left\{ - \int_0^t d\tau [c \mu(0+, 0|0, +c; \tau) \right. \\ \left. + 2c \mu(0+, -c|0, +c; \tau)] \right\}, \end{aligned} \quad (9)$$

$$S^R(0; t) = \exp \left\{ - \int_0^t d\tau c \mu(0+, -c|0, 0; \tau) \right\}, \quad (10)$$

and the two-particle conditional probability density obeys the following closed equation [5]:

$$\begin{aligned} \left\{ \frac{\partial}{\partial t} + z v_{21} + \frac{\dot{S}^R(v_1; t)}{S^R(v_1; t)} - \frac{\dot{S}^R(v_2; t)}{S^R(v_2; t)} \right\} \tilde{\mu}(z, v_2|v_1; t) - v_{21} \mu(0+, v_2|0, v_1; t) \\ = \int dv_3 \int dv_4 \tilde{\mu}(z, v_3|v_1; t) \tilde{\mu}(z, v_2|v_4; t) v_{34} \theta(v_{34}) \mu(0+, v_4|0, v_3; t), \end{aligned} \quad (11)$$

where the dot stands for the time derivative, $v_{21} = v_2 - v_1$, and we use the definition

$$\tilde{\mu}(z, u|v; t) = \int_{0+}^{\infty} dx e^{-zx} \mu(x, u|0, v; t). \quad (12)$$

It should be stressed that this remarkable property of getting a rigorous closed equation for the distribution μ comes from the main characteristic of the annihilation process: only those particles which have not interacted could survive until time t . For more details see Ref. [5]. We can now start to solve Eq. (11) for our symmetric three-velocity case. Equation (11) can be rewritten in a matrix form:

$$\dot{\mathcal{N}}(z; t) + \mathcal{C}(z; t) = \mathcal{N}(z; t) \mathcal{C}(z; t) \mathcal{N}(z; t), \quad (13)$$

where \mathcal{N} and \mathcal{C} are two 3×3 matrices:

$$\mathcal{N}(z; t) = (N_{ij}(z; t))_{i,j=1,2,3}, \quad (14)$$

with

$$\begin{aligned} N_{11}(z; t) &= \tilde{\mu}(z, +c|+c; t), \\ N_{12}(z; t) &= \tilde{\mu}(z, 0|+c; t) e^{-zct} S^R(+c; t) / S^R(0; t), \\ N_{13}(z; t) &= \tilde{\mu}(z, -c|+c; t) e^{-2zct} S^R(+c; t), \\ N_{21}(z; t) &= \tilde{\mu}(z, +c|0; t) e^{zct} S^R(0; t) / S^R(+c; t), \\ N_{22}(z; t) &= \tilde{\mu}(z, 0|0; t), \\ N_{23}(z; t) &= \tilde{\mu}(z, -c|0; t) e^{-zct} S^R(0; t), \\ N_{31}(z; t) &= \tilde{\mu}(z, +c|-c; t) e^{2zct} / S^R(+c; t), \\ N_{32}(z; t) &= \tilde{\mu}(z, 0|-c; t) e^{zct} / S^R(0; t), \\ N_{33}(z; t) &= \tilde{\mu}(z, -c|-c; t), \end{aligned} \quad (15)$$

and

$$\mathcal{C}(z; t) = \begin{pmatrix} 0 & C_{12}(z; t) & C_{13}(z; t) \\ 0 & 0 & C_{23}(z; t) \\ 0 & 0 & 0 \end{pmatrix}, \quad (16)$$

with

$$\begin{aligned} C_{12}(z; t) &= c \mu(0+, 0|0, +c; t) e^{-zct} S^R(+c; t) / S^R(0; t), \\ C_{13}(z; t) &= 2c \mu(0+, -c|0, +c; t) e^{-2zct} S^R(+c; t), \\ C_{23}(z; t) &= c \mu(0+, -c|0, 0; t) e^{-zct} S^R(0; t). \end{aligned} \quad (17)$$

The initial values of \mathcal{N} and \mathcal{C} are

$$\mathcal{N}(z; 0) = \frac{\sigma}{z + \sigma} \begin{pmatrix} p_+ & p_0 & p_+ \\ p_+ & p_0 & p_+ \\ p_+ & p_0 & p_+ \end{pmatrix}, \quad (18)$$

$$\mathcal{C}(z; 0) = \begin{pmatrix} 0 & c\sigma p_0 & 2c\sigma p_+ \\ 0 & 0 & c\sigma p_+ \\ 0 & 0 & 0 \end{pmatrix}. \quad (19)$$

In spite of the fact that Eq. (13) looks very simple, it should be noticed that it is a matrix differential equation for which the general solution is not known. However, the particular structure of \mathcal{C} makes it solvable. The method of solution is described in Appendix A. We find

$$\begin{aligned} N_{11}(z; t) &= \frac{p_+ \sigma}{D(z; t)} \left(1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{23}(z; \tau) \right]^2 \right), \\ N_{12}(z; t) &= \frac{p_0 \sigma}{D(z; t)} \left(1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{23}(z; \tau) \right]^2 \right) \\ &\quad - \frac{p_0}{p_+} \int_0^t d\tau C_{23}(z; \tau), \\ N_{13}(z; t) &= \frac{p_+ \sigma}{D(z; t)} \left(1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{23}(z; \tau) \right]^2 \right) \\ &\quad - \int_0^t d\tau C_{13}(z; \tau), \\ N_{21}(z; t) &= \frac{p_+ \sigma}{D(z; t)}, \\ N_{22}(z; t) &= \frac{p_0 \sigma}{D(z; t)}, \\ N_{23}(z; t) &= \frac{p_+ \sigma}{D(z; t)} \left(1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{23}(z; \tau) \right]^2 \right) \\ &\quad - \int_0^t d\tau C_{23}(z; \tau), \\ N_{31}(z; t) &= \frac{p_+ \sigma}{D(z; t)}, \\ N_{32}(z; t) &= \frac{p_0 \sigma}{D(z; t)}, \\ N_{33}(z; t) &= \frac{p_+ \sigma}{D(z; t)} \left(1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{2,3}(z; \tau) \right]^2 \right), \end{aligned} \quad (20)$$

where

$$\begin{aligned} D(z; t) &= z + \sigma - 2p_0 \sigma \int_0^t d\tau C_{23}(z; \tau) \\ &\quad - p_+ \sigma \int_0^t d\tau C_{13}(z; \tau). \end{aligned} \quad (21)$$

The Laplace transformed conditional probabilities $\tilde{\mu}$ are obtained from Eqs. (15). Thus we found the exact solution to the kinetic equation (11). However, this is as yet an implicit solution, that is to say, the values of $\tilde{\mu}$ are expressed in terms of the density at contact $\mu(0+, u|0, v; t)$ ($u < v$). To obtain some information about physical quantities, we have to solve the consistency equations which express the three nonvanishing densities at contact in terms of function $\tilde{\mu}$:

$$\mu(0+, u|0, v; t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2i\pi} \tilde{\mu}(z, u|v; t) \quad (u < v). \quad (22)$$

Up to this point, the generalization to a nonsymmetric three-velocity distribution is straightforward.

III. LONG TIME BEHAVIOR OF THE SYSTEM

Instead of writing the consistency equations for the density at contact, we shall write them for the survival probabilities. Remembering that only two densities at contact are independent [see Eq. (6)] and that [from Eqs. (9) and (10)]

$$\dot{S}^R(0; t) = -c\mu(0, -c|0, 0; t) S^R(0; t), \quad (23)$$

$$\begin{aligned} \dot{S}^R(+c; t) &= [-c\mu(0, 0|0, +c; t) - 2c\mu(0, -c|0, +c; t)] \\ &\quad \times S^R(+c; t), \end{aligned} \quad (24)$$

we obtain, using Eq. (22), the following two equations:

$$\dot{S}^R(0; t) = -cp_+ \sigma \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2i\pi} e^{zct} \frac{1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{23}(z; \tau) \right]^2}{D(z; t)}, \quad (25)$$

$$\dot{S}^R(+c; t) = \frac{p_0}{p_+} S^R(0; t) \dot{S}^R(0; t) - 2cp_+\sigma \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2i\pi} e^{2zct} \frac{\left(1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{23}(z; \tau)\right]^2\right)^2}{D(z; t)}. \quad (26)$$

Solving these two equations seems to be extremely difficult, principally because of the time dependence in $D(z; t)$ and in the integrals over τ . However, it can be shown (see Appendix B) that the only relevant time dependence occurs in the exponential factor. By this we mean that Eqs. (25) and (26) can be exactly replaced by the following two relations:

$$\dot{S}^R(0; t) = -cp_+\sigma \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2i\pi} e^{zct} \frac{1 - \frac{p_0}{2p_+} \left[\int_0^\infty d\tau C_{23}(z; \tau)\right]^2}{D(z; \infty)}, \quad (27)$$

$$\dot{S}^R(+c; t) = \frac{p_0}{p_+} S^R(0; t) \dot{S}^R(0; t) - 2cp_+\sigma \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2i\pi} e^{2zct} \frac{\left(1 - \frac{p_0}{2p_+} \left[\int_0^\infty d\tau C_{23}(z; \tau)\right]^2\right)^2}{D(z; \infty)}. \quad (28)$$

This remarkable fact allows us to use again the Laplace transform to suppress the inverse Laplace integral and arrive at much simpler relations:

$$\begin{aligned} T(z) &\equiv \int_0^\infty dt e^{-zct} \dot{S}^R(0; t) \\ &= -p_+\sigma \frac{1 - \frac{p_0}{2p_+} T^2(z)}{z + \sigma + 2p_0\sigma T(z) + p_+\sigma U(z)}, \end{aligned} \quad (29)$$

$$\begin{aligned} U(z) &\equiv \int_0^\infty dt e^{-2zct} \left[\dot{S}^R(+c; t) - \frac{p_0}{p_+} \dot{S}^R(0; t) S^R(0; t) \right] \\ &= -p_+\sigma \frac{\left[1 - \frac{p_0}{2p_+} T^2(z)\right]^2}{z + \sigma + 2p_0\sigma T(z) + p_+\sigma U(z)}, \end{aligned} \quad (30)$$

where Eqs. (17) and (21) have been used. From this system of equations it follows that $T(z)$ satisfies the quartic equation

$$\begin{aligned} p_0 T^4(z) - (2p_0 + 1) T^2(z) - 2 \left(\frac{z}{\sigma} + 1\right) T(z) + p_0 - 1 \\ = 0, \end{aligned} \quad (31)$$

$U(z)$ being given by

$$U(z) = T(z) - \frac{p_0}{2p_+} T^3(z). \quad (32)$$

It is hopeless to try to Laplace invert the solution of Eq. (31) to find $\dot{S}^R(0; t)$. However, the asymptotic regime of the survival probability can be determined by considering $T(z)$ for z in the neighborhood of 0. For $z = 0$, one knows that $T(0)$ is given by

$$T(0) = \int_0^\infty dt \dot{S}^R(0; t) = S^R(0, \infty) - 1 \quad (33)$$

and that $S^R(0, \infty) \in [0, 1]$. This condition allows us to identify the physically acceptable solution of the above quartic equation (31). We have to distinguish between two different situations.

(i) $p_0 \leq \frac{1}{4}$. The only acceptable solution is $T(0) = -1$, corresponding to an asymptotically empty stationary state $S^R(+c; \infty) = S^R(0; \infty) = 0$.

(ii) $p_0 \geq \frac{1}{4}$. We have two possible solutions: $T(0) = -1$ and $T(0) = 1 - \frac{1}{\sqrt{p_0}}$. However, $S^R(0; t)$ is a continuous decreasing function of time with initial value 1. Thus the stationary value $2 - \frac{1}{\sqrt{p_0}}$ will be reached first. Accordingly, $T(0) = 1 - \frac{1}{\sqrt{p_0}}$ is the relevant value.

We can now study the behavior of $T(z)$ near zero, by introducing the quantity $\epsilon(z) \equiv T(z) - T(0)$ in the quartic equation (31). Five cases have to be distinguished.

(i) $p_0 = 0$. This is the bimodal velocity distribution already investigated [5,1]. The quartic equation (31) simplifies to the following quadratic equation for $U(z)$:

$$U^2(z) + 2 \left(\frac{z}{\sigma} + 1\right) U(z) + 1 = 0. \quad (34)$$

By solving and Laplace inverting it, one recovers the Elskens and Frisch results [1].

(ii) $0 < p_0 < \frac{1}{4}$. Now $\epsilon(z) = T(z) + 1$ obeys the quartic equation:

$$p_0 \epsilon^4(z) - 4p_0 \epsilon^3(z) + (4p_0 - 1) \epsilon^2(z) - 2 \frac{z}{\sigma} [\epsilon(z) - 1] = 0. \quad (35)$$

As we are interested in the limit $z \rightarrow 0$, and as $\lim_{z \rightarrow 0} \epsilon(z) = 0$, we can neglect the terms of order ϵ^4 and ϵ^3 with respect to ϵ^2 , and ϵ with respect to 1. By Laplace inverting and integrating, we obtain the following asymptotic behavior ($t \rightarrow \infty$):

$$S^R(0; t) = \sqrt{\frac{2}{(1 - 4p_0)\pi c\sigma t}} \left\{ 1 + O\left([1 - 4p_0]^{-3} [c\sigma t]^{-1}\right) \right\}, \quad (36)$$

$$S^R(+c; t) = \frac{1}{1-p_0} \sqrt{\frac{1-4p_0}{\pi c \sigma t}} \left\{ 1 + O\left([1-4p_0]^{-3/2} [c \sigma t]^{-1/2}\right) \right\}. \quad (37)$$

For $t \rightarrow \infty$, the densities are then given by

$$\sigma(0; t) = \frac{2p_0}{(1-4p_0)c \pi t} \left\{ 1 + O\left([1-4p_0]^{-3} [c \sigma t]^{-1}\right) \right\}, \quad (38)$$

$$\sigma(+c; t) = \sqrt{\left(\frac{1}{4} - p_0\right) \frac{\sigma}{c \pi t}} \left\{ 1 + O\left([1-4p_0]^{-3/2} [c \sigma t]^{-1/2}\right) \right\}. \quad (39)$$

The corrections to scaling have been obtained by a careful study of the solution of the quartic equation (31).

The prefactor $(4p_0 - 1)$ in the ϵ^2 term in Eq. (35) indicates that for p_0 near $\frac{1}{4}$ we should expect an important crossover effect. This is confirmed by the amplitude of the correction to scaling for $\sigma(0; t)$ which contains a term proportional to $(1 - 4p_0)^{-4}$. As a consequence, it may be very difficult to extract the true asymptotic behavior from experimental or numerical data. However, we see that for very long times the system is driven towards the bimodal case.

(iii) $p_0 = \frac{1}{4}$. Here again, $\epsilon(z) = T(z) + 1$. The coefficient of the ϵ^2 term vanishes and the term of order ϵ^3 cannot be neglected. We are thus left with the cubic equation:

$$\epsilon^3(z) - 2 \frac{z}{\sigma} = 0 \quad \text{when } z \rightarrow 0 \quad (40)$$

whose physically relevant solution leads to the asymptotic behavior (for $t \rightarrow \infty$):

$$S^R(0; t) = \frac{1}{\Gamma(2/3)} \left(\frac{2}{c \sigma t}\right)^{1/3} \left\{ 1 + O\left([c \sigma t]^{-1/3}\right) \right\}, \quad (41)$$

$$S^R(+c; t) = 2 \left[\left(\frac{2^{1/3}}{3\Gamma(2/3)}\right)^2 + \frac{2}{\Gamma(1/3)} \right] \left(\frac{1}{c \sigma t}\right)^{2/3} \left\{ 1 + O\left([c \sigma t]^{-1/3}\right) \right\}. \quad (42)$$

For the density one gets ($t \rightarrow \infty$)

$$\sigma(0; t) = \frac{\sigma}{4\Gamma^2(2/3)} \left(\frac{2}{c \sigma t}\right)^{2/3} \left\{ 1 + O\left([c \sigma t]^{-1/3}\right) \right\}, \quad (43)$$

$$\sigma(+c; t) = \frac{3\sigma}{8} \left[\frac{1}{3} \left(\frac{2^{1/3}}{\Gamma(2/3)}\right)^2 + \frac{1}{\Gamma(1/3)} \right] \left(\frac{1}{c \sigma t}\right)^{2/3} \left\{ 1 + O\left([c \sigma t]^{-1/3}\right) \right\}. \quad (44)$$

In addition,

$$\lim_{t \rightarrow \infty} \frac{\sigma(0; t)}{\sigma(+c; t)} = \frac{1}{2} + \frac{3}{2} \frac{\Gamma^2(2/3)}{2^{2/3} \Gamma(1/3)} \simeq 1.15. \quad (45)$$

(iv) $\frac{1}{4} < p_0 < 1$. Now $\epsilon(z) = T(z) - 1 + \frac{1}{\sqrt{p_0}}$. The terms of order ϵ^4 and ϵ^3 can be neglected with respect to the term ϵ^2 and variable z with respect to 1. Hence

$$\epsilon(z) = \frac{2\sqrt{p_0} - 1}{p_0(5 - 2\sqrt{p_0})} - \sqrt{\frac{2(1 - \sqrt{p_0})}{p_0(2\sqrt{p_0} - 1)(5 - 2\sqrt{p_0})}} \left[\frac{(2\sqrt{p_0} - 1)^3}{2(1 - \sqrt{p_0})(5 - 2\sqrt{p_0})} + \frac{z}{\sigma} \right]^{1/2}, \quad (46)$$

which leads to an exponential asymptotic behavior for $S^R(0; t) - 2 + \frac{1}{\sqrt{p_0}}$ and also for $S^R(+c; t)$. Thus, for $t \rightarrow \infty$

$$\sigma(0; t) \simeq \sigma (2\sqrt{p_0} - 1)^2 + 2\sigma A \frac{e^{-c \sigma t}}{(c \sigma t)^{3/2}}, \quad (47)$$

$$\sigma(+c; t) \simeq \sigma A \frac{e^{-c \sigma t}}{(c \sigma t)^{3/2}}, \quad (48)$$

where

$$A = \sqrt{\frac{2p_0(1 - \sqrt{p_0})(5 - 2\sqrt{p_0})}{\pi(2\sqrt{p_0} - 1)}} \frac{1 - \sqrt{p_0}}{(2\sqrt{p_0} - 1)^2},$$

$$u = \frac{(2\sqrt{p_0} - 1)^3}{2p_0(1 - \sqrt{p_0})(5 - 2\sqrt{p_0})}.$$

In addition,

$$\lim_{t \rightarrow \infty} \frac{\sigma(0; t) - \sigma(0; \infty)}{\sigma(+c; t)} = 2, \quad (49)$$

indicating that in the long time regime the collisions between pairs of moving particles disappear, and only collisions involving a particle at rest and a moving one can be observed.

(v) $p_0 = 1$. In this trivial limiting case, in which there are no collisions, the physical solution is $T(z) \equiv 0$, and the density remains constant.

As anticipated on heuristic arguments, we have shown that $p_0 = \frac{1}{4}$ is playing a special role separating the different kinetic regimes.

Another important quantity that can be computed exactly concerns the correlations between the velocities of the colliding particles. Although at time $t = 0$ these velocities are uncorrelated, annihilation dynamics creates strong correlations between them during the time evolution. This has the important consequence of excluding a Boltzmann-like approximation. It is clearly seen in $\bar{w}(v; t)$, the mean velocity of the right nearest neighbor of a particle moving with velocity v :

$$\bar{w}(v; t) = c\bar{\mu}(0, +c|v; t) - c\bar{\mu}(0, -c|v; t). \quad (50)$$

Using Eqs. (20) we obtain, for example,

$$\bar{w}(-c; t) = \frac{c}{D(0; t)} \left[p_+ S^R(+c; t) - p_+ + \frac{p_0}{2} \left(\int_0^t d\tau C_{23}(0; \tau) \right)^2 \right],$$

with $D(0; t)$ given by Eq. (21) and $C_{23}(0; t)$ by Eq. (17). Our results show that pairs of nearest neighbors have the tendency to align their velocities and propagate in the same direction. Indeed, in the limit $t \rightarrow \infty$, one finds for $p_0 < \frac{1}{4}$

$$\bar{w}(v; \infty) = \begin{cases} \pm c & \text{if } v = \pm c \\ c(\sqrt{2} - 1) \frac{1 - 4p_0}{1 - 2p_0} & \text{if } v = 0, \end{cases} \quad (51)$$

for $p_0 = \frac{1}{4}$

$$\bar{w}(v; \infty) = v, \quad (52)$$

and for $p_0 > \frac{1}{4}$

$$\bar{w}(v; \infty) = v(p_0^{-1/2} - 1). \quad (53)$$

IV. GENERAL DISCRETE VELOCITY DISTRIBUTION

In this section, we want to consider the case of a general discrete velocity distribution. Suppose that the initial velocity distribution is given by

$$\phi(v; 0) = \sum_{k=1}^n p_k \delta(v - v_k), \quad (54)$$

with

$$\sum_{k=1}^n p_k = 1.$$

As for the three-velocity case, we assume initially random spatial distribution and no correlations between the velocities. We can generalize some of our formulas in a straightforward way: the probability for the absence of collisions with the right neighbor is given by

$$S^R(v; t) = \exp \left\{ \int_0^t d\tau \sum_{k=1}^n (v - v_k) \theta(v - v_k) \times \mu(0+, v_k|0, v; \tau) \right\}. \quad (55)$$

If we define ($i, j = 1, \dots, n$)

$$N_{ij}(z; t) = \tilde{\mu}(z, v_j|v_i; t) \exp \left\{ -zv_{ij}t + \int_0^t d\tau \sum_{k=1}^3 [B_{ik}(\tau) - B_{jk}(\tau)] \right\}, \quad (56)$$

$$C_{ij}(z; t) = B_{ij}(t) \exp \left\{ -zv_{ij}t + \int_0^t d\tau \sum_{k=1}^3 [B_{ik}(\tau) - B_{jk}(\tau)] \right\}, \quad (57)$$

where

$$B_{ij}(t) = v_{ij} \theta(v_{ij}) \mu(0, v_j|0, v_i; t),$$

the evolution equation for the Laplace transformed conditional probability $\tilde{\mu}$ [Eq. (11)] can be put in the same form as Eq. (13), but with an $n \times n$ matrix, we find $\mathcal{A}(z) = \mathcal{N}_3(z; 0)$ and $\dot{\mathcal{B}}(z; t) = \mathcal{E}(z; t)$. Once again, the particular structure of the matrix \mathcal{C} will help us in solving this equation: assuming that $\mathcal{N}(z; t) - \mathcal{I}$ is invertible (this can be explicitly checked for $t = 0$ and verified *a posteriori* for later times), where \mathcal{I} is the $n \times n$ unit matrix, we define

$$\mathcal{P}(z; t) = (\mathcal{N}(z; t) - \mathcal{I})^{-1} + \frac{\mathcal{I}}{2}. \quad (58)$$

Then, Eq. (13) reads

$$\dot{\mathcal{P}}(z; t) = -\mathcal{P}(z; t)\mathcal{C}(z; t) - \mathcal{C}(z; t)\mathcal{P}(z; t). \quad (59)$$

One way to solve this equation is to remember that $C_{ij}(z; t) = 0$ for $i \geq j$ ($i, j = 1, \dots, n$). Hence, Eq. (59) has the following structure:

$$-\dot{P}_{ij}(z; t) = \sum_{k=1}^{j-1} P_{ik} C_{kj} + \sum_{k=i+1}^n C_{ik} P_{kj} \quad (60)$$

and thus when $i = n$, $j = 1$ it takes the form

$$\dot{P}_{n,1}(z; t) = 0.$$

In addition, the $(n-1, 1)$ and $(n, 2)$ equations are

$$\begin{aligned} -\dot{P}_{n-1,1}(z;t) &= C_{n-1,n}(z;t)P_{n,1}(z;t), \\ -\dot{P}_{n,2}(z;t) &= C_{1,2}(z;t)P_{n,1}(z;t), \end{aligned}$$

and can be solved once $P_{n,1}(z;t)$ is known. One can easily see that the equations for $(n-2, 1)$, $(n-1, 2)$, and $(n, 3)$ are expressed in terms of $P_{n,1}$, $P_{n-1,1}$, and $P_{n,2}$, allowing us to complete the solution. The process is iterated until solving equation $(1, n)$ and determining entirely the matrix $\mathcal{P}(z;t)$. The n^2 Laplace transformed conditional probabilities are obtained straightforwardly by inverting the formula (58). Again we have an implicit solution and we have to solve the $n(n-1)/2$ consistency equations [see Eq. (22)] to obtain physical information on the system. Although detailed calculations can be very tedious, there should not be any conceptual difficulties.

V. CONCLUDING REMARKS

We have shown in this paper that exact predictions on ballistic-annihilation kinetics can be obtained in the framework of the approach described in [5]. It is remarkable that, for discrete velocity distributions, the nonlinear integrodifferential equation governing the two-particle conditional probability can be solved exactly.

For $p_0 \leq \frac{1}{4}$, not only are the leading power laws obtained in the asymptotic regime, $t \rightarrow \infty$, but also the amplitudes and the corrections to scaling occur when p_0 is in the neighborhood of $\frac{1}{4}$. For $p_0 > \frac{1}{4}$, exponential behavior is obtained for long times. The results of our numerical simulations and those of Redner and co-workers [3,4] are well explained by our exact theory. In addition, it is clearly shown that Boltzmann-like approximations fail, because the annihilation dynamics favors configurations in which the nearest neighbors have the same velocity.

Several extensions of this work are possible. In particular, the case of continuous velocity distributions for which qualitatively different behavior may be expected is under investigation.

ACKNOWLEDGMENTS

M.D. was supported by the Swiss National Foundation. The work of L.F. was partly done in Geneva and was supported by the Swiss National Foundation. J.P. acknowledges the hospitality of the Department of Theoretical Physics of the University of Geneva.

APPENDIX A: ON THE MATRIX EQUATION

The solution of the matrix equation (13) is presented here. To solve it, we can write \mathcal{N} as

$$\mathcal{N}(z;t) = \begin{pmatrix} \mathcal{N}_1(z;t) & \mathcal{N}_2(z;t) \\ \mathcal{N}_3(z;t) & \mathcal{N}_4(z;t) \end{pmatrix}, \quad (\text{A1})$$

where \mathcal{N}_1 is a 1×2 matrix, \mathcal{N}_2 a 1×1 , \mathcal{N}_3 a 2×2 , and \mathcal{N}_4 a 2×1 . In this decomposition, \mathcal{C} is given by

$$\mathcal{C}(z;t) = \begin{pmatrix} \mathcal{E}_1(z;t) & \mathcal{E}_2(z;t) \\ 0 & \mathcal{E}_3(z;t) \end{pmatrix}, \quad (\text{A2})$$

where

$$\begin{aligned} \mathcal{E}_1(z;t) &= (0 \quad C_{12}(z;t)) , \\ \mathcal{E}_2(z;t) &= (C_{13}(z;t)) , \\ \mathcal{E}_3(z;t) &= \begin{pmatrix} C_{23}(z;t) \\ 0 \end{pmatrix} . \end{aligned}$$

However, to perform the matrix product $\mathcal{N}\mathcal{C}\mathcal{N}$, we have to write \mathcal{C} in a different way, namely,

$$\mathcal{C}(z;t) = \begin{pmatrix} 0 & \mathcal{E}(z;t) \\ 0 & 0 \end{pmatrix}, \quad (\text{A3})$$

where \mathcal{E} is a 2×2 matrix

$$\mathcal{E}(z;t) = \begin{pmatrix} C_{1,2}(z;t) & C_{1,3}(z;t) \\ 0 & C_{2,3}(z;t) \end{pmatrix} .$$

With this decomposition, the matrix equation (13) becomes

$$\begin{pmatrix} \dot{\mathcal{N}}_1(z;t) & \dot{\mathcal{N}}_2(z;t) \\ \dot{\mathcal{N}}_3(z;t) & \dot{\mathcal{N}}_4(z;t) \end{pmatrix} + \begin{pmatrix} \mathcal{E}_1(z;t) & \mathcal{E}_2(z;t) \\ 0 & \mathcal{E}_3(z;t) \end{pmatrix} = \begin{pmatrix} \mathcal{N}_1(z;t)\mathcal{E}(z;t)\mathcal{N}_3(z;t) & \mathcal{N}_1(z;t)\mathcal{E}(z;t)\mathcal{N}_4(z;t) \\ \mathcal{N}_3(z;t)\mathcal{E}(z;t)\mathcal{N}_3(z;t) & \mathcal{N}_3(z;t)\mathcal{E}(z;t)\mathcal{N}_4(z;t) \end{pmatrix}. \quad (\text{A4})$$

Although \mathcal{N}_3 is not invertible, we can find a solution for the equation

$$\dot{\mathcal{N}}_3(z;t) = \mathcal{N}_3(z;t)\mathcal{E}(z;t)\mathcal{N}_3(z;t) \quad (\text{A5})$$

using the following ansatz:

$$\mathcal{N}_3(z;t) = \mathcal{A}(z) [\mathcal{I} - \mathcal{B}(z;t)\mathcal{A}(z)]^{-1}, \quad (\text{A6})$$

where \mathcal{I} is the unit 2×2 matrix, we find $\mathcal{A}(z) = \mathcal{N}_3(z;0)$

and $\dot{\mathcal{B}}(z;t) = \mathcal{E}(z;t)$. Once we have obtained \mathcal{N}_3 , we can readily solve the three other matrix equations and thus find \mathcal{N} .

APPENDIX B: ON THE CONSISTENCY EQUATIONS

In this appendix we show how to justify Eqs. (27) and (28) starting from Eqs. (25) and (26).

A way of demonstration could be to simply subtract Eqs. (27) and (25) and to verify by explicit integration in the complex plane that the result is zero. Instead of going through all this tedious algebra, we prefer to give a simple physical argument.

We define $\tilde{A}_1(z; t)$ by

$$\tilde{\mu}(z, +c | +c; t) = p_+ \sigma \tilde{A}_1(z; t). \quad (\text{B1})$$

From the kinetic equation (11), we have

$$\begin{aligned} \dot{\tilde{\mu}}(z, +c | +c; t) &= \sum_{i=1}^3 \sum_{j=1}^3 \tilde{\mu}(z, v_i | +c; t) \tilde{\mu}(z, +c | v_j; t) \\ &\quad \times v_{ij} \theta(v_{ij}) \mu(0+, v_j | 0, v_i; t). \end{aligned} \quad (\text{B2})$$

Now, the right hand side of Eq. (B2) represents the variation of $\tilde{\mu}$ due to the mutual annihilation of particles originally separating the pair $(+c, +c)$ [see Ref. [5] for the interpretation of different terms in Eq. (11)]. It follows that the inverse Laplace transform

$$A_1(x; t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2i\pi} e^{xz} \tilde{A}_1(z; t) \quad (\text{B3})$$

represents the probability that the particles present initially in an interval of length x , separating two particles with velocity $+c$, disappear through ballistic annihilation before time t [the prefactor $p_+ \sigma$ in Eq. (B1) is the initial density of the right neighbors moving with velocity $+c$]. As the colliding pairs move at least with a relative velocity c , all sequences of encounters contributing to $A_1(x; t)$ are accomplished at the moment $t^* = x/c$. Therefore $A_1(x; t)$ does not depend on time (for fixed x) in the region $t > t^*$.

Now we remember that

$$\tilde{A}_1(z; t) = \frac{1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{23}(z; \tau) \right]^2}{D(z; t)}, \quad (\text{B4})$$

hence Eq. (25) reads

$$\dot{S}^R(0; t) = -cp_+ \sigma \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2i\pi} e^{zct} \tilde{A}_1(z; t). \quad (\text{B5})$$

In the term on the right hand side of Eq. (B5), the inverse Laplace transformation yields the value of function $A_1(x; t)$ at the point $x = ct$. So we can replace $\tilde{A}_1(z; t)$ in Eq. (B5) by $\tilde{A}_1(z; \infty)$, without changing the value of the integral. Hence we obtain Eq. (27).

Equation (26) can be handled in the same manner: we first define

$$\begin{aligned} p_+ \sigma \tilde{A}_2(z; t) &= \tilde{\mu}(z, -c | +c; t) e^{-2zct} S^R(+c; t) \\ &\quad + \int_0^t dt C_{13}(z; t). \end{aligned} \quad (\text{B6})$$

Then, by inspecting the evolution equation for $\tilde{\mu}(z, -c | +c; t)$ we remark that $A_2(x; t)$, the inverse Laplace transform of $\tilde{A}_2(z; t)$, represents the probability that all particles present initially in an interval of length x , separating one particle of velocity $+c$ on the left and one of velocity $-c$ on the right, disappear through ballistic annihilation before time t . It follows that $A_2(x; t)$ does not depend on the time (for fixed x) in the region $t > t^*/2$ (as the relative velocity between the colliding pairs must be $2c$). Finally, the expression

$$\tilde{A}_2(z; t) = \frac{\left(1 - \frac{p_0}{2p_+} \left[\int_0^t d\tau C_{23}(z; \tau) \right]^2 \right)^2}{D(z; t)} \quad (\text{B7})$$

leads to the expected conclusion.

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